# Generating Parity Relations for Detecting and Identifying Control System Component Failures

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**Generalized parity relations of an especially simple form for monitoring control system sensors and actuators for failures are defined. They are based on a discrete-time model of the dynamics of linear, time-invariant systems. In the past, generalized parity relations have been constructed by operating on the state space description of the system dynamics. It is shown here that an alternate approach based on a transfer matrix description of the system is very useful for deriving the parity relations and interpreting their properties. An original result is a method for constructing the parity relation of minimum length that depends on the output of only one sensor.**

## Introduction

FAULT tolerance has become an essential feature of control systems in many application areas. Aircraft flight control systems are now often designed with redundant sets of components managed in a way that allows the system to continue to perform its function in spite of the failure of some of its components. Control systems for nuclear power plants are often specified to have reliability levels that can only be achieved through fault tolerance. An area of special current interest is the control of large structures in space, where the combination of many sensors and actuators required to allow the system to perform its function, with the long operating time desired between visits for maintenance and resupply, virtually assures that many components will fail during the operating period.

The traditional approach to achieving fault tolerance has been though the use of hardware replication. If, for example, each required sensor is triplicated or quadruplicated, the outputs of like sensors can be compared directly, and if one output deviates significantly from the others, that sensor can be declared failed. More recently, interest has shifted to the use of *analytic redundancy* rather than massive hardware redundancy. In the space structure control application, for example, if the system requires 100 sensors to sense adequately the vibrations of the structure, it would be unreasonable to ask to triplicate each sensor. Analytic redundancy depends on a model of the dynamics of the controlled system to relate actuator commands and sensor outputs, thus enabling checks for consistency without comparing the outputs of replicated components directly. An unfortunate consequence of this approach to failure detection and identification (FDI) is that errors in the representation of the system dynamics confuse the detection and isolation of component failures.

Many methods of performing the FDI function have been suggested. Willsky[15] gives a summary of the major concepts that have been exploited. Some of these concepts are applicable only to sensor FDI whereas others can be applied to both sensor and actuator failure detection and identification. Another distinction among these methods is that some depend, for their design and operation, on hypotheses about the modes of component failure whereas others do not. Among all these methods, two are distinguished as being applicable to both sensor and actuator FDI in addition to not requiring assumptions about how the failed component behaves. These are the methods of generalized parity relations, first studied by Chow,[2] and the failure detection filter introduced by Beard[1] which was later amplified by Jones[7] and recently re-examined by Massoumnia.[11]

In this paper we suggest a particular configuration of generalized parity relations for sensor and actuator failure detection that makes the isolation of the failed component trivial. If the actuators are considered fully reliable, the identification of the failed sensor succeeds, even in the presence of multiple simultaneous sensor failures. These parity relations are interpreted in terms of a state space model of the system dynamics and from the point of view of an input–output transfer matrix. The minimum possible length for the parity relation is also determined. Moreover, a simple algorithm for the construction of these parity relations is given, and some examples are shown. In most cases, it is possible to construct parity relations of this type for actuator failures as well, and that subject is also discussed. Additional examples of them are given by Dutilloy,[5] who also illustrates their performance in the presence of noise and modeling error.

## FDI System Configuration

The usual configuration of a failure detection and isolation system is shown in Fig. 1. The residual generation function is the part of the processor under consideration in this paper. The residual generator takes as inputs the commands to the system actuators and the sensor outputs, and puts out a set of residuals, which are monitored for evidence of failures. The monitoring of the residuals is done by the failure decision processor, which usually involves threshold tests and simple logic.

In order to be useful as indicators of component failures, the residuals should be small in the absence of failures, and one or more of them should become large in the presence of a failure. Moreover, the set of residuals that becomes large should be indicative of which component failed. The first requirement is met by constructing the residuals to be independent of the system state under nominal conditions. In the absence of failures, then, the residuals are only due to unmodeled noise and disturbances. When a failure occurs, this relationship is violated, and one or more of the residuals becomes larger.

The second requirement implies that the failure of each control system component should cause a distinguishably different combination of residuals to grow large. The sets of residuals that indicate the failure of different sensors or actu-

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ators can be formulated in any number of ways (see Ref. 12 for a few examples). The simplest and probably the most reliable coding strategy to isolate sensor failures is to have just one residual associated with each sensor—so that the failure of any sensor affects just one residual, and the residual that is affected can be associated with just one sensor. In addition to its obvious simplicity, the reliability of this approach results from the fact that the successful identification of a failed sensor solely depends upon a single residual exceeding a threshold, whereas alternative approaches depend not only on most of the residuals remaining small but also upon more than one of them becoming significantly large. Thus, our intent is to construct a set of generalized parity relations which are independent of the state of the system and each one of which depends on the output of one and only one sensor. Typically, each such sensor parity relation also depends on the performance of all of the actuators in the system.

A complete FDI system usually includes another set of residuals designed to detect actuator failures. Those residuals can possibly be generated by another set of parity relations, each of which is associated with a single actuator; they also typically depend on all of the sensors as well. So the configuration of the FDI system is as indicated in Fig. 2. Upon the failure of a sensor, just one of the sensor residuals and all, or at least most, of the actuator residuals would become large. The logic in the failure decision block would decide that one sensor had failed rather than that most of the actuators had simultaneously failed. Similarly, the failure of one actuator would be easily distinguished from the simultaneous failure of most of the sensors.

It should be noted that the described coding of residuals is not necessarily unambiguous, since, for example, it may turn out that the failure of the $i$th sensor only affects the $j$th actuator parity relation instead of most of the actuator parity relations, and that at the same time the failure of the $j$th actuator only affects the $i$th sensor parity relation instead of most of the sensor parity relations. Clearly, with such residuals it would be impossible to say whether the $i$th sensor or the $j$th actuator has failed. However, such special cases are rare in practice, and we do not consider them here. We refer the reader to Ref. 12 for a thorough discussion of the closely related concept of failure identifiability.

## Single-Sensor Parity Relations

Let us consider the discrete linear time-invariant (LTI) system

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

with $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^l$. A failure of the $i$th sensor can be modeled by rewriting Eq. (2) as

$$y(t) = Cx(t) + e_i m_i(t) \tag{3}$$

where $e_i$ is the $i$th column of the $l \times l$ identity matrix and $m_i(t)$ is some unknown arbitrary function of time. For example, if the $i$th sensor fails dead with zero output, then this failure mode corresponds to $m_i(t) = -c_i' x(t)$ where $c_i'$ is the $i$th row of $C$. Similarly, if the $i$th sensor has a bias $b$, then $m_i(t) = b$. Note that this type of failure modeling does not depend on the mode of sensor failure and is quite general.

Using the described model and assuming for the time being that the actuators are fully reliable, one can generate a very simple residual by forming a linear combination of the finite past and present outputs of a single sensor. This combination is chosen to be zero when the sensor is functioning properly but nonzero when the sensor fails. We call this form of residual a single-sensor parity relation (SSPR). SSPRs are special cases of the generalized parity relations discussed in Chow[2,3] and Lou.[10]

To illustrate the idea, assume that we are at time $t+s$ and that we combine, with appropriate weightings, the measurements of the $i$th sensor from the past time $t$ up to the present time $t+s$. Using the known dynamics of the system and assuming that the actuators are perfectly reliable,

$$\begin{bmatrix} y_i(t) \\ y_i(t+1) \\ \vdots \\ y_i(t+s) \end{bmatrix} = \begin{bmatrix} c_i' \\ c_i'A \\ \vdots \\ c_i'A^s \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ c_i'B & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_i'A^{s-1}B & c_i'A^{s-2}B & \cdots & c_i'B \end{bmatrix} \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+s-1) \end{bmatrix} \tag{4}$$

We can rewrite Eq. (4) as

$$\Gamma_s \bar{u}(t) - \bar{y}_i(t) = -P_s x(t) \tag{5}$$

where $\bar{y}_i(t) = [y_i(t), y_i(t+1), \ldots, y_i(t+s)]'$, $\bar{u}(t) = [u'(t), u'(t+1), \ldots, u'(t+s-1)]'$, and $\Gamma_s$ and $P_s$ have an obvious correspondence with the matrices in Eq. (4). A single-sensor parity relation $r_i(t+s)$ is simply defined as

$$r_i(t+s) := \alpha'[\Gamma_s \bar{u}(t) - \bar{y}_i(t)] \tag{6}$$

where $\alpha'$ is some row vector such that $\alpha' P_s = 0$ (cf. Ref. 2). Note that for an appropriately large $s$, it follows from the Cayley-Hamilton theorem that such $\alpha$ always exists. Now using the definition of $\alpha$, it is clear that $r_i(t+s)$ is zero when the sensor is functioning properly, but in the presence of a failure in the $i$th sensor, this residual may become nonzero; hence it can be used to detect and identify the failure of the $i$th sensor. (Recall that for the moment the actuators are assumed to be fully reliable.) Let the components of $\alpha$ be as follows:

$$\alpha = [\alpha_0, \alpha_1, \ldots, \alpha_{s-1}, 1]' \tag{7}$$

For normalization purposes and without the loss of generality, we have set the last component of $\alpha$ to 1. Now let $q$ be the forward shift operator [i.e., $qu(t) = u(t+1)$] and consequently rewrite Eq. (6) as

$$q^s r_i(t) = \left[ \sum_{j=1}^{s} c_i' \psi_j(A) B q^{j-1} \right] u(t) - \psi_0(q) y_i(t) \tag{8}$$

where

$$\begin{aligned} \psi_0(q) &= q^s + \alpha_{s-1} q^{s-1} + \cdots + \alpha_1 q + \alpha_0 \\ \psi_1(q) &= q^{s-1} + \alpha_{s-1} q^{s-2} + \cdots + \alpha_1 \\ &\vdots \\ \psi_{s-1}(q) &= q + \alpha_{s-1} \\ \psi_s(q) &= 1 \end{aligned}$$

Clearly, the polynomials $\psi_j(q)$ satisfy the backward recursion

$$\psi_{j-1}(q) = \psi_j(q) q + \alpha_{j-1}, \qquad (j \in s), \qquad \psi_s(q) = 1 \tag{9}$$

Note that the elements of the vector $\alpha$ are the only unknowns in Eq. (8). Also, the length of the window $s$ has not yet been specified. Of particular interest are those parity relations for which the length of the window is minimal. We refer to these residuals as the minimum-length SSPR (also see

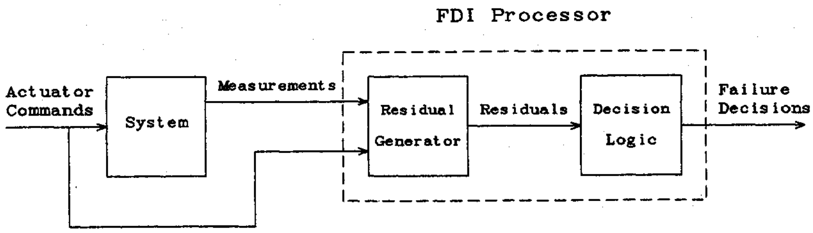


Fig. 1 Typical configuration of FDI system.

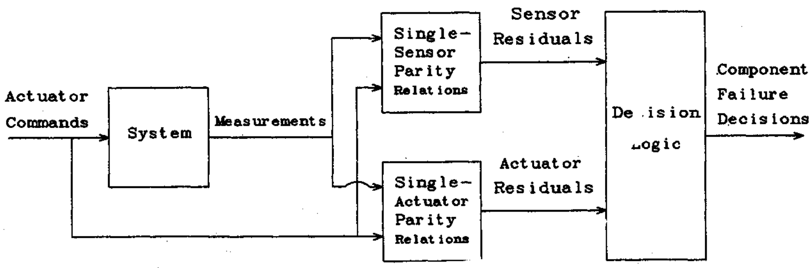


Fig. 2 FDI system based on single-component parity relations.

Ref. 10). Interestingly enough, this problem has a very simple solution. We can rewrite $\alpha' P_s = 0$ as $c_i' \psi_0(A) = 0$. Consequently, the polynomial $\psi_0(q)$ is simply the minimal annihilating polynomial of $c_i'$ with respect to $A$ (see Chapter 5 of Ref. 6).

This fact can be restated in more familiar terms if we change the basis by an appropriate similarity transformation. Let us define the transformation $z(t) = Tx(t)$ where $T := [Q', P_{s-1}']'$ with $P_{s-1}$ as before and $Q$ any matrix such that $T$ is nonsingular. Note that when $s$ is minimal, the rows of $P_{s-1}$ are linearly independent, and the last row of $P_s$ is a linear combination of the rows of $P_{s-1}$. In the new basis, the transformed matrix $A_t = TAT^{-1}$ and the transformed measurement vector $c_{it}' = c_i' T^{-1}$ will have the following structure:

$$A_t = \begin{bmatrix} A_1 & A_2 \\ 0 & A_0 \end{bmatrix}$$

$$c_{it}' = \begin{bmatrix} 0 & c_0' \end{bmatrix} \tag{10}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{s-1} \end{bmatrix}$$

$$c_0' = [1 \quad 0 \quad 0 \cdots \quad 0] \tag{11}$$

It is clear that the pair $(c_0', A_0)$ is observable and that the polynomial $\psi_0(q)$ is simply the characteristic polynomial of $A_0$. In other words, the minimal annihilating polynomial of $c_i'$ is the product of the terms $(q - \lambda_j)$, where $\lambda_j$ are the eigenvalues corresponding to those modes of $A$ observable from $c_i'$.

This interpretation provides us with a numerically reliable procedure for computing the coefficients of the polynomial $\psi_0(q)$. One only needs to find the observable modes of $(c_i', A)$ using a numerically reliable algorithm (see Refs. 9, 14, 13). One of the simplest solutions is to choose a random $n$ vector $d_1$ and compute $\sigma_0 = \sigma(A)$ and $\sigma_1 = \sigma(A + d_1 c_i')$; [$\sigma(A)$ is the spectrum of the matrix $A$]. The unobservable spectrum $\sigma_{uo}$ almost surely consists of the set of the common elements of $\sigma_0$ and $\sigma_1$. Let $\sigma_{ob} = \sigma_0 - \sigma_{uo}$; then

$$\psi_0(q) = \prod_{\lambda \in \sigma_{ob}} (q - \lambda) \tag{12}$$

Knowing $\psi_0(q)$, we can compute $c_i' \psi_j(A)$ [see Eq. (8)] using the backward recursion in Eq. (9). We only need to compute $c_i' \psi_j(A)$ and not the computationally more expensive terms $\psi_j(A)$. If in the process of computing the unobservable modes the unobservable subspace $S$ of $(c_i', A)$ is also found, then construct a full-row rank matrix $P$ such that the null space of $P$ is equal to $S$ and let $A_0 = PAP^{-r}$ and $c_0' = c_i' P^{-r}$ where $P^{-r}$ is a right-inverse of $P$. Now we can use the factor system $(c_0', A_0)$ in place of $(c_i', A)$ in Eq. (8) to simplify the computation of the coefficients of the parity relation. Note that the coefficients of the minimum-length SSPR do not depend on the particular basis used for computing them and are invariant under similarity transformation.

We also point out that the residual in Eq. (8) is simply the innovation of a deadbeat observer that reconstructs the portion of the state space that is not in the unobservable subspace $S$. In other words, to find the minimum-length SSPR for the $i$th sensor, simply factor out that part of the state space that is unobservable from the $i$th sensor and then construct a deadbeat observer for the remainder of the state space. The innovation of this observer is the residual that we are looking for. The relation between these parity relations and the residual generators proposed by Clark[4] (see also Ref. 12) should be obvious.

Now we rederive the SSPR using an input-output approach. This approach will be useful later for deriving the single-actuator parity relations.

Using Eq. (3), we can write the output of the $i$th sensor as

$$y_i(t) = m_i(t) + \frac{\phi_i'(q)}{\psi(q)} Bu(t) \tag{13}$$

where $\phi_i(q) \in R^n[q]$ and $\psi(q)$ are a coprime factorization of $c_i'(qI - A)^{-1}$ and $m_i(t)$ is an arbitrary unknown scalar function representing the effect of the failure. Reordering Eq. (13), we have

$$m_i(t) = y_i(t) - \frac{\phi_i'(q)}{\psi(q)} Bu(t) \tag{14}$$

Now we generate the residual $r_i(t)$ by filtering $m_i(t)$ through any one-to-one linear system satisfying certain stability requirements. In order to assign the dynamics of the residual generator to arbitrary locations inside the unit circle, we simply take

$$r_i(t) = -\frac{\mu(q)\psi(q)}{\omega(q)} m_i(t) \tag{15}$$

where $\mu(q)$ is any arbitrary polynomial (*which should be set equal to a constant for the minimum-length parity relation*) and $\omega(q)$ is any desired stable polynomial with an order at least as large as the order of $\mu(q)\psi(q)$. When the residual is generated as in Eq. (15), the transfer vector relating the effect of the initial condition $x(0)$ on the residual $r_i(t)$ is proportional to $\mu(q)\phi_i'(q)/\omega(q)$, which is stable. Substituting Eq. (14) into Eq. (15), we have

$$r_i(t) = \frac{\mu(q)\phi_i'(q)}{\omega(q)} Bu(t) - \frac{\mu(q)\psi(q)}{\omega(q)} y_i(t) \tag{16}$$

If we choose $\omega(q) = q^s$ for some appropriate integer $s$, the residual generator will exhibit a deadbeat response. The rational function coefficients of $y_i(t)$ and $u(t)$ in Eq. (16) can be rewritten as polynomials in the backward shift operator $q^{-1}$, i.e., the residual generator will be a finite impulse response (FIR) filter.

Using the definition of $\psi(q)$ and $\phi_i'(q)$ we have

$$[\mu(q)\phi_i'(q), -\mu(q)\psi(q)] \begin{bmatrix} qI - A \\ c_i' \end{bmatrix} = 0 \tag{17}$$

Hence the parity relation is simply a polynomial vector in the left null space of the singular pencil $P(q) = [c_i, qI - A']'$. This interpretation of a parity vector is discussed in detail in Refs. 10 and 12.

It is simple to show that the polynomial $\psi(q)$ in Eq. (16) is the same as the minimal annihilating polynomial $\psi_0(q)$ we defined earlier in this section. Moreover, the reader should note the relation between the recursive polynomials in Eq. (9) and the method of Faddeeva[6,8] for computing the adjoint of $qI - A$.

**Example 1**

Let us consider the system in Eqs. (1) and (2) with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Going through the details of the outlined procedure, it can be shown that for the first sensor

$$\psi_0(q) = q^2 - 2q$$

Substituting the preceding equation into Eq. (8), it follows that

$$r_1(t) = -2u_1(t-2) + u_1(t-1) + u_2(t-1) - y_1(t) + 2y_1(t-1)$$

‡Given an arbitrary vector $w(t)$, we denote the $i$th component of $w(t)$ by $w_i(t)$.

Repeating the procedure for the second sensor, we get

$$\psi_0(q) = q - 1$$

Substituting this into Eq. (8), it follows that

$$r_2(t) = u_1(t-1) + u_2(t-1) - y_2(t) + y_2(t-1)$$

Assuming that the actuators are fully reliable, the first residual is only affected by the failure of the first sensor, and the second residual is only affected by the failure of the second sensor. Hence these two residuals can be used to identify any sensor failure.

In the following section, we generate single-actuator parity relations. Assuming that the sensors are fully reliable, these parity relations can be used to identify even simultaneous actuator failures. However, contrary to the SSPR, it is not always possible to generate the single-actuator parity relations, and there is a solvability condition that we shall derive.

## Single-Actuator Parity Relations

Considering the system in Eqs. (1) and (2), let us assume that the effect of actuator failures can be modeled by rewriting the dynamics as

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Lm(t) \\ y(t) &= Cx(t) \end{aligned} \tag{18}$$

where $m(t)$ is some unknown function of time identically equal to zero when the actuators are functioning properly. Each element of $m(t)$ corresponds to an actuator that is to be monitored for failure; let there be $k$ of them. Each column of $L$ is a column of $B$ corresponding to a monitored actuator. If the performances of all of the actuators are to be observed, then $L = B$.

Suppose that the first column of $L$ and the first element of $m(t)$ correspond to actuator number one. Then if the first actuator fails dead and does not respond to the command, the effect of this failure can be modeled by taking the first component of $m(t)$ as the negative of the first component of $u(t)$. If the first actuator has a bias of magnitude $b$, then $m_1(t) = b$.‡ Therefore, the representation in Eq. (18) can be used to model the effect of a wide variety of actuator failure modes.

In addition, let us assume that the pair $(C, A)$ is observable and rewrite Eq. (18) as

$$y(t) = G_u(q)u(t) + G_m(q)m(t) \tag{19}$$

where $G_u(q) := C(qI - A)^{-1}B$ and $G_m(q) := C(qI - A)^{-1}L$. As mentioned in the introduction, the objective of single-actuator parity relations (SAPR) is to generate a set of residuals such that when no failure is present all the residuals are zero or close to zero, and in the presence of the $i$th actuator failure only the residual corresponding to actuator $i$ becomes distinguishably nonzero.

This objective can be restated in terms of transfer matrices as generating a $k$-dimensional residual vector $r(t)$ by passing the observation vector $z(t) = [y'(t), u'(t)]'$ through a causal linear time-invariant system characterized by the transfer matrix $H(q)$, i.e.,

$$r(t) = H(q)z(t) = [H_y(q), H_u(q)] \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \tag{20}$$

such that the net transmission from the input $u(t)$ to the residual vector $r(t)$ is zero and the failure mode $m_i(t)$ only affects the $i$th component of the residual vector $r(t)$. In other words, the objective is to find a proper postcompensator $H(q)$ such that

$$H(q)G(q) = [-T(q),0] \tag{21}$$

where the 0 in Eq. (21) is a $k \times m$ matrix

$$G(q) = \begin{bmatrix} G_m(q) & G_u(q) \\ 0 & I \end{bmatrix} \tag{22}$$

and $T(q)$ is a $k \times k$ diagonal matrix with the nonzero diagonal elements $T_i(q)$.

In addition, when no failure is present, the effect of the initial mismatch between the state of the residual generator and the state of the system should die away so that the residual vector $r(t)$ stays close to zero. The residual due to a nonzero initial condition $x(0)$ is simply $qH_y(q)G_s(q)x(0)$ where

$$G_s(q) := C(qI-A)^{-1} \tag{23}$$

Hence, the transfer matrix $H_y(q)G_s(q)$ should be stable. Also, the residual due to the nonzero initial conditions of the postcompensator should die away, so we require that $H(q)$ be stable.

The problem we have formulated has a very simple solution in terms of transfer matrices.

*Theorem*: It is possible to generate SAPR if and only if the transfer matrix $G_m(q)$ is left invertible.

*Proof*: If and only if we can generate a set of SAPR, then there exists an $H_y(q)$ such that $H_y(q)G_m(q) = -T(q)$. But $T(q)$ is by definition full-column rank; hence $G_m(q)$ should be full-column rank or equivalently left invertible.

Let us denote the left inverse of $G_m(q)$ by $G_m^{-l}(q)$. Using Eq. (19), we have

$$m(t) = G_m^{-l}(q)y(t) - G_m^{-l}(q)G_u(q)u(t) \tag{24}$$

To generate the residual $r(t)$, we pass $-m(t)$ through a diagonal filter $T(q)$ with the nonzero diagonal elements $T_i(q)$. It is clear that by the appropriate selection of the numerators and denominators of $T_i(q)$, it is possible to arbitrarily assign the dynamics of the proper transfer matrices

$$H_y(q) := -T(q)G_m^{-l}(q) \tag{25}$$

and $H_y(q)G_s(q)$. Moreover, let

$$H_u(q) := T(q)G_m^{-l}(q)G_u(q) \tag{26}$$

Clearly, the stability of $H_y(q)G_s(q)$ implies that $H_u(q)$ given in Eq. (26) is stable. Hence all the requirements are satisfied, and $H(q) := [H_y(q), H_u(q)]$ is the residual generator that we are looking for.

If deadbeat dynamics are assigned to the residual generator described in the Theorem, i.e., if the poles of $H(q)$ are assigned to the origin of the complex plane, then the resulting residuals are the parity relations described in Refs. 3 and 5. Moreover, if in addition to the observability of $(C, A)$ the pair $(A, L)$ is also controllable, then the selection of the diagonal elements $T_i(q)$ is particularly simple. In this case, set the numerator of $T_i(q)$ to the least common multiple of the denominators of the elements of the $i$th row of $G_m^{-l}(q)$ and set the denominator of $T_i(q)$ to any stable polynomial with a degree such that the $i$th row of $H_y(q)$ in Eq. (25) is proper. [Note that the nonminimum phase zeros of $G_m(q)$ will automatically show up in the numerators of $T(q)$.] Using this procedure, the transfer matrices $H_y(q)$ and $H_y(q)G_m(q)$ are clearly stable. Using the Lemma (see Appendix) and the controllability of $(A, L)$, it follows immediately that $H_y(q)G_s(q)$ is also stable. The next step is to construct $H_u(q)$ according to Eq. (26). Clearly, the residual generator $H(q)$ constructed in this fashion satisfies all the necessary requirements. We now illustrate the outlined design procedure by continuing Example 1.

*Example 2*

Let us monitor both actuators in Example 1 for possible failure; then $L = B$ and $G_m(q) = G_u(q)$. A simple computation shows that for this example

$$G_u(q) = \begin{bmatrix} \dfrac{1}{q} & \dfrac{1}{q-2} \\ \dfrac{1}{q-1} & \dfrac{1}{q-1} \end{bmatrix}$$

The inverse of $G_u(q)$ is simply

$$G_u^{-1}(q) = \begin{bmatrix} -.5q(q-2) & .5q(q-1) \\ .5q(q-2) & -.5(q-1)(q-2) \end{bmatrix}$$

Let us choose $T_i(q) = 1/q^2$ (for deadbeat response). Then using Eq. (25),

$$H_y(q) = \begin{bmatrix} .5-q^{-1} & -.5+.5q^{-1} \\ -.5+q^{-1} & .5-1.5q^{-1}+q^{-2} \end{bmatrix}$$

Also using Eq. (26),

$$H_u(q) = \begin{bmatrix} q^{-2} & 0 \\ 0 & q^{-2} \end{bmatrix}$$

Translating back to the time domain,

$$r_1(t) = u_1(t-2) + 0.5y_1(t) - y_1(t-1) - 0.5y_2(t) + 0.5y_2(t-1)$$

$$r_2(t) = u_2(t-2) - 0.5y_1(t) + y_1(t-1) + 0.5y_2(t) - 1.5y_2(t-1) + y_2(t-2)$$

Assuming that the sensors are fully reliable, $r_1(t)$ is only affected by the failure of the first actuator and $r_2(t)$ is only affected by the failure of the second actuator. Hence these two residuals can be used to identify any actuator failure.

We also point out that if we set $G_m(q)$ in Eq. (19) equal to the identity matrix, then this is clearly equivalent to modeling the effect of sensor failures, and the design procedure given in the Theorem would be equivalent to the procedure for generating the single-sensor parity relations outlined at the end of the third section. The identity matrix is obviously left invertible; that is why it is always possible to construct the single-sensor parity relations for all the sensors.

## Conclusions

The construction of generalized parity relations for monitoring control system sensors and actuators for failure signatures has been discussed in the context of both state space and transfer matrix descriptions of the dynamics of the system. Parity relations of a form that permits especially easy identification of the faulty sensor or actuator have been defined. One group of these residual generators has the property that each relation depends only on a single sensor—and on all the actuators in general. They are called *single-sensor* parity relations. Another group, called *single-actuator* parity relations, has the opposite property; each one depends on only a single actuator—and on all the sensors in general.

These parity relations can be generated by performing linear operations on the state space description of the discrete-time dynamics of the controlled system. It has been shown that an alternate approach, using the transfer matrix characterization of the system, affords an easy derivation of the form of the parity relations and a computational procedure for generating them. The single-sensor parity relation of minimum length is defined, and the solvability condition is given for a group of single-actuator parity relations.

## Appendix

*Lemma*: Let $(C, A, L)$ be a minimal realization of $G_m(q)$, and assume that we have a stable postcompensator $H_y(q)$ for which $H_y(q)G_m(q)$ is stable. Then it follows that $H_y(q)G_s(q)$ is stable where $G_s(q) := C(qI - A)^{-1}$.

*Proof*: Let $D_l^{-1}(q)\Psi(q)$ be a left coprime factorization (cf. Ref. 8) of $G_s(q)$. Also, let $N_h(q)D_h^{-1}(q)$ be a right coprime factorization of $H_y(q)$. Using these definitions, $H_yG_s = N_h(D_lD_h)^{-1}\Psi$ and $H_yG_m = N_h(D_lD_h)^{-1}\Psi L$ (to simplify the notation we have deleted the argument $q$). To prove the stability of $H_yG_s$ using the stability of $H_yG_m$, we have to show that any possible cancellation between $D_lD_h$ and $\Psi L$ is a stable cancellation, since the polynomial matrices $D_lD_h$ and $\Psi$ are left coprime and have only unimodular common factors. Because $(A, L)$ is assumed to be controllable, the polynomial matrices $D_l$ and $\Psi L$ are left coprime, and using the generalized Bezout identity (Lemma 6.3-9 of Ref. 8), we know

$$\begin{bmatrix} -N_r & X^* \\ D_r & Y^* \end{bmatrix}\begin{bmatrix} -X & Y \\ D_l & \Psi L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{A1}$$

for the appropriate matrices $N_r$, $D_r$, $X$, $Y$, $X^*$, and $Y^*$. (Note that all three block matrices in Eq. (A1) are unimodular.) Multiplying both sides of Eq. (A1) from the right by the block diagonal matrix $diag\{D_h, I\}$, we get

$$\begin{bmatrix} -N_r & X^* \\ D_r & Y^* \end{bmatrix}\begin{bmatrix} -XD_h & Y \\ D_lD_h & \Psi L \end{bmatrix} = \begin{bmatrix} D_h & 0 \\ 0 & I \end{bmatrix} \tag{A2}$$

Now let us denote the greatest common left divisor of $D_l(q)D_h(q)$ and $\Psi(q)L$ by $Q(q)$. We know there exists a unimodular matrix $U(q)$ (with block partitions $U_{11}$, $U_{12}$, $U_{21}$, and $U_{22}$) such that (see Lemma 6.3-3 of Ref. 8):

$$[D_lD_h, \Psi L]\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = [Q, 0] \tag{A3}$$

Multiplying both sides of Eq. (A2) by $U$ and using Eq. (A3), we have

$$\begin{bmatrix} -N_r & X^* \\ D_r & Y^* \end{bmatrix}\begin{bmatrix} M_1 & M_2 \\ Q & 0 \end{bmatrix} = \begin{bmatrix} D_h & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \tag{A4}$$

for the appropriate matrices $M_1$ and $M_2$. Using the unimodular property of the block matrices in the right- and left-hand side of Eq. (A4), it follows immediately that

$$\det[Q(q)] \times \det[M_2(q)] = \text{const} \times \det[D_h(q)]$$

Also, the stability of $H_y(q)$ implies that $\det[D_h(q)] = 0$ has stable roots. Hence the roots of $\det[Q(q)] = 0$ are stable, and using the stability of $H_yG_m$, it follows that $H_yG_s$ is stable.

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